

Criteria for Morin singularities for maps into lower dimensions, and applications

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Abstract

We give criteria for Morin singularities for germs of maps into lower dimensions. As an application, we study the bifurcation of Lefschetz singularities.

1 Introduction

A map-germ $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ ($m > n$) is called a k -Morin singularity ($1 \leq k \leq n$) if it is \mathcal{A} -equivalent to the following map-germ at the origin:

$$(1.1) \quad \begin{aligned} & h_{0,k}(x_1, \dots, x_{n-1}, y_1, \dots, y_{m-n}, z) \\ &= \left(x_1, \dots, x_{n-1}, q(y_1, \dots, y_{m-n}) + z^{k+1} + \sum_{i=1}^{k-1} x_i z^i \right) \end{aligned}$$

if $k \geq 2$, and $h_{0,1}(x_1, \dots, x_{n-1}, y_1, \dots, y_{m-n+1}) = (x_1, \dots, x_{n-1}, q(y_1, \dots, y_{m-n+1}))$ if $k = 1$, where q is a non-degenerate quadratic germ of function. The 1-Morin singularity is also called the *fold*, and the 2-Morin singularity is also called the *cusp*. We say that two map-germs $f, g : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ are \mathcal{A} -equivalent if there exist germs of diffeomorphism $\varphi : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^m, 0)$ and $\Phi : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ such that $\Phi \circ f \circ \varphi = g$. Morin singularities are stable, and conversely, all corank one and stable map-germs are Morin singularities. This means that Morin singularities are fundamental and frequently appear as singularities of maps from one manifold to another. If $\text{corank } df_0 = 1$, then one can choose a coordinate system (x, y) such that $f(x, y) = (x_1, \dots, x_{n-1}, h(x, y))$, where $x = (x_1, \dots, x_{n-1})$, $y = (y_1, \dots, y_{m-n+1})$. We call this procedure a *normalization*. Morin [17] gave a characterization of those singularities in terms of transversality of the jet extensions to the Thom-Boardman singularity set, and also gave criteria for germs with respect to a normalized form $(x_1, \dots, x_{n-1}, h(x, y))$. Morin singularities are also characterized using the intrinsic derivative due to Porteous ([20] see also [1, 7]). Criteria for singularities without using normalization are not only more convenient but also indispensable in some cases. We refer to criteria which are independent of normalization as *general criteria*. In fact, in the case of wave front surfaces in 3-space, general criteria for cuspidal edges and swallowtails were given in [14], where we used them to study the local and global behavior of flat fronts in hyperbolic 3-space. Recently general criteria for other singularities and several of their applications have been given in [11, 12, 13, 19, 26, 27, 28]. In this paper, we give general criteria for Morin singularities. Using them, we give applications to bifurcation of the Lefschetz singularity which plays important roles in low-dimensional topology. See [5, 6, 10, 21, 22, 23, 24, 29, 30] for other investigations of Morin singularities.

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2 Singular sets and Hesse matrix of corank one singularities

Definition 2.1. Let $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ be a map-germ and denote by $S(f)$ the singular locus of f . A collection of vector fields

$$(2.1) \quad (\xi, \eta) = (\xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_{m-n+1})$$

on $(\mathbf{R}^m, 0)$ is said to be *adapted with respect to f* if $\xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_{m-n+1}$ generates $T_0 \mathbf{R}^m$ at 0, and $\langle \eta_1(p), \dots, \eta_{m-n+1}(p) \rangle_{\mathbf{R}} = \ker df_p$ for any $p \in S(f)$ near 0.

Lemma 2.2. *Let $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ be a map-germ satisfying $\text{rank } df_0 = n - 1$. Then there exists a collection of vector fields (ξ, η) as in (2.1) which is adapted with respect to f .*

Proof. Since the result does not depend on the choice of coordinate system and $\text{rank } df_0 = n - 1$, then we can take a coordinate system $(x, y) = (x_1, \dots, x_{n-1}, y_1, \dots, y_{m-n+1})$ in a neighborhood of the origin U on the source space, such that

$$(2.2) \quad f(x, y) = (x, h(x, y)), \quad dh_0 = 0.$$

Then $S(f) = \{(x, y) \in U \mid h_{y_1}(x, y) = \dots = h_{y_{m-n+1}}(x, y) = 0\}$ holds. Thus $\partial x_1, \dots, \partial x_{n-1}, \partial y_1, \dots, \partial y_{m-n+1}$ are the desired vector fields. \square

Let $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ be a map-germ satisfying $\text{rank } df_0 = n - 1$, and $(\xi, \eta) = (\xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_{m-n+1})$ an adapted collection of vector fields with respect to f . Set

$$\lambda_i = \det(\xi_1 f, \dots, \xi_{n-1} f, \eta_i f), \quad i = 1, \dots, m - n + 1$$

and

$$\Lambda = (\lambda_1, \dots, \lambda_{m-n+1}),$$

where ζf stands for the directional derivative of f along the vector field ζ . Then $S(f) = \{\Lambda = 0\}$.

Definition 2.3. Let 0 be a singular point of $f = (f_1, \dots, f_n) : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ satisfying $\text{rank } df_0 = n - 1$. We say that 0 is *non-degenerate* if $\text{rank } d\Lambda_0 = m - n + 1$.

This condition is a special case of the condition called *critical normalization*. See [4] for details.

Lemma 2.4. *The non-degeneracy condition above does not depend on the choice of coordinate systems on the source space nor on the target space.*

Proof. One can easily show that it does not depend on the coordinate system on the target. In fact, let $\Phi : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a germ of diffeomorphism, and we regard $d\Phi_x$ as the matrix representation of $d\Phi_x$ with respect to the standard basis at $x \in \mathbf{R}^n$. Set $\bar{\lambda}_i = \det(\xi_1(\Phi \circ f), \dots, \xi_{n-1}(\Phi \circ f), \eta_i(\Phi \circ f))$, and $\bar{\Lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{m-n+1})$. Then $\bar{\Lambda}(x) = \det(d\Phi_{f(x)})\Lambda(x)$ holds. Thus $\text{rank } d\Lambda_0$ does not depend on the choice of the coordinate system on the target.

Secondly, we show that it does not depend on the choice of an adapted collection of vector fields. Since it does not depend on the coordinate system on the target, we may assume that $f = (f_1, \dots, f_n)$ satisfies $d(f_n)_0 = 0$. Then for any vector field ζ , it holds that $\zeta \lambda_i = \det(\xi_1 f, \dots, \xi_{n-1} f, \zeta \eta_i f)(0) = (\Delta \zeta \eta_i f_n)(0)$, where $\Delta = \det(\xi_1 \hat{f}, \dots, \xi_{n-1} \hat{f})$, and

$\hat{f} = (f_1, \dots, f_{n-1})$. Let $(\bar{\xi}_1, \dots, \bar{\xi}_{n-1}, \bar{\eta}_1, \dots, \bar{\eta}_{m-n+1})$ be an adapted collection of vector fields satisfying

$$(2.3) \quad \begin{pmatrix} \bar{\xi}_1 \\ \vdots \\ \bar{\xi}_{n-1} \\ \bar{\eta}_1 \\ \vdots \\ \bar{\eta}_{m-n+1} \end{pmatrix} = \left(\begin{array}{c|c} A^1 & A^2 \\ \hline B^1 & B^2 \end{array} \right) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n-1} \\ \eta_1 \\ \vdots \\ \eta_{m-n+1} \end{pmatrix}, \quad \text{where} \quad \left(\begin{array}{c|c} A^1 & A^2 \\ \hline B^1 & B^2 \end{array} \right) = \begin{pmatrix} a_{1,1}^1 & \cdots & a_{1,n-1}^1 & a_{1,1}^2 & \cdots & a_{1,m-n+1}^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1}^1 & \cdots & a_{n-1,n-1}^1 & a_{n-1,1}^2 & \cdots & a_{n-1,m-n+1}^2 \\ \hline b_{1,1}^1 & \cdots & b_{1,n-1}^1 & b_{1,1}^2 & \cdots & b_{1,m-n+1}^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{m-n+1,1}^1 & \cdots & b_{m-n+1,n-1}^1 & b_{m-n+1,1}^2 & \cdots & b_{m-n+1,m-n+1}^2 \end{pmatrix},$$

where A^1, B^2 are regular matrices at 0, and $B^1 = O$ on $S(f)$. Set

$$(2.4) \quad \bar{\lambda}_i = \det(\bar{\xi}_1 f, \dots, \bar{\xi}_{n-1} f, \bar{\eta}_i f) \quad (i = 1, \dots, m-n+1), \quad \bar{\Lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{m-n+1}).$$

Then for any vector field ζ , we see that

$$\zeta \bar{\lambda}_i(0) = (\det A^1 \Delta \zeta \bar{\eta}_i f_n)(0),$$

and $d\bar{\Lambda}_0 = ((\det A^1 \Delta) B^2 d\Lambda)(0)$. Thus we have the conclusion. \square

For a non-degenerate singularity 0, we define a matrix \mathcal{H}_η by

$$(2.5) \quad \mathcal{H}_\eta = (\eta_j \lambda_i)_{1 \leq i, j \leq m-n+1} = \begin{pmatrix} \eta_1 \lambda_1 & \cdots & \eta_1 \lambda_{m-n+1} \\ \vdots & \ddots & \vdots \\ \eta_{m-n+1} \lambda_1 & \cdots & \eta_{m-n+1} \lambda_{m-n+1} \end{pmatrix}.$$

Then \mathcal{H}_η is symmetric on $S(f)$. In fact, since $[\eta_j, \eta_i](p) \in T_p \mathbf{R}^m$, there exist functions α_i ($i = 1, \dots, n-1$) and β_j ($j = 1, \dots, m-n+1$) such that

$$(2.6) \quad [\eta_j, \eta_i](p) = \sum_{i=1}^{n-1} \alpha_i \xi_i(p) + \sum_{j=1}^{m-n+1} \beta_j \eta_j(p).$$

If $p \in S(f)$, then by $\eta_j f(p) = 0$ and (2.6) it follows that

$$\eta_j \lambda_i = \det(\xi_1 f, \dots, \xi_{n-1} f, \eta_j \eta_i f) = \det(\xi_1 f, \dots, \xi_{n-1} f, \eta_i \eta_j f) = \eta_i \lambda_j$$

on $S(f)$.

Lemma 2.5. *Let 0 be a non-degenerate singular point of $f = (f_1, \dots, f_n) : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$. The matrix-valued function \mathcal{H}_η on $S(f)$ does not depend on the choice of an adapted collection of vector fields with respect to f , nor on the coordinate systems on the target up to non-zero functional multiplications. In particular, $\text{rank } \mathcal{H}_\eta$ on $S(f)$ does not depend on the choice of adapted collections of vector fields with respect to f nor on the coordinate systems on the target.*

Proof. Let $(\xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_{m-n+1})$ and $(\bar{\xi}_1, \dots, \bar{\xi}_{n-1}, \bar{\eta}_1, \dots, \bar{\eta}_{m-n+1})$ be adapted collections of vector fields with respect to f satisfying (2.3). By the conditions, it holds that $B^1 = 0$, $\det A^1 \neq 0$ and $\det B^2 \neq 0$ on $S(f)$. Set

$$\bar{\lambda}_i = \det(\bar{\xi}_1 f, \dots, \bar{\xi}_{n-1} f, \bar{\eta}_i f), \quad \mathcal{H}_{\bar{\eta}} = (\bar{\eta}_j \bar{\lambda}_i)_{1 \leq i, j \leq m-n+1}.$$

Since η_i and $\bar{\eta}_i$ ($i = 1, \dots, m-n+1$) are included in $\ker df$ on $S(f)$, one can see that:

$$\begin{aligned} \bar{\eta}_j \bar{\lambda}_i &= \bar{\eta}_j \det(\bar{\xi}_1 f, \dots, \bar{\xi}_{n-1} f, \bar{\eta}_i f) \\ &= \langle \bar{\xi}_1 f \times \dots \times \bar{\xi}_{n-1} f, \bar{\eta}_j \bar{\eta}_i f \rangle \\ &= \det A^1 \langle \xi_1 f \times \dots \times \xi_{n-1} f, \bar{\eta}_j \bar{\eta}_i f \rangle \\ &= \det A^1 \left\langle \xi_1 f \times \dots \times \xi_{n-1} f, \sum_{k,l} \eta_l b_{i,k}^1 \xi_l f + \sum_{k,l} b_{j,l}^2 b_{i,k}^2 \eta_l \eta_k f \right\rangle \\ &= \det A^1 \sum_{k,l} b_{j,l}^2 b_{i,k}^2 \langle \xi_1 f \times \dots \times \xi_{n-1} f, \eta_l \eta_k f \rangle \\ &= \det A^1 \sum_{k,l} b_{j,l}^2 b_{i,k}^2 \eta_l \lambda_k. \end{aligned}$$

Thus on $S(f)$, we have that

$$(2.7) \quad \mathcal{H}_{\bar{\eta}} = (\det A^1)^{m-n+1} (\det B^2) \mathcal{H}_{\eta}.$$

This proves the first assertion. One can show the independence for the target coordinate systems easily by following the same method as used in the proof of Lemma 2.4. \square

If 0 is a non-degenerate singularity, then $S(f)$ is a manifold. Thus we can consider $g = f|_{S(f)}$. Then we have the following lemma.

Lemma 2.6. *Let 0 be a non-degenerate singular point of $f = (f_1, \dots, f_n)$. Then*

$$S(f|_S) = S(g) = \{p \in S(f) \mid \det \mathcal{H}_{\eta}(p) = 0\}$$

near 0. Moreover, by the identification

$$(2.8) \quad \mathcal{H}_{\eta} : \sum_{i=1}^{m-n+1} a_i \eta_i \mapsto \sum_{i=1}^{m-n+1} \left(\sum_{j=1}^{m-n+1} a_j \eta_i \lambda_j \right) \eta_i,$$

it holds that $\ker dg_p = \ker \mathcal{H}_{\eta}(p) = \ker df_p \cap T_p S(f)$.

Proof. The assumption and results do not depend on the choice of coordinate systems, so we may assume that f has the form (2.2). Let us assume that $\text{rank Hess}_0 h(0, y) = k$. Then by the parametrized Morse Lemma (see [9, p.502], [2, p.97]), there exist a coordinate system $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_{m-n+1})$ and a function \tilde{h} such that

$$\begin{aligned} h(x, y) &= q(\tilde{y}) + \tilde{h}(x, \tilde{y}_{k+1}, \dots, \tilde{y}_{m-n+1}), \\ q(\tilde{y}) &= \sum_{i=1}^k e_i \tilde{y}_i^2, \quad \tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_k), \quad e_i = \pm 1 \end{aligned}$$

holds. We rewrite the coordinate as $(y_1, \dots, y_k) = (\tilde{y}_1, \dots, \tilde{y}_k)$ and $z = (z_1, \dots, z_{k'}) = (\tilde{y}_{k+1}, \dots, \tilde{y}_{m-n+1})$, where $k' = m-n+1-k$. Then $f(x, y, z) = (x, f_n(x, y, z))$ has the form

$$(2.9) \quad f_n(x, y, z) = q(y) + \tilde{\tilde{h}}(x, z), \quad q(y) = \sum_{i=1}^k e_i y_i^2, \quad \text{Hess } \tilde{\tilde{h}}(0, z)(0) = 0.$$

We rewrite $\tilde{h}(x, z) = h(x, z)$. Furthermore, by Lemma 2.5, one can take an adapted collection of vector fields

$$\xi_i = \partial x_i \ (i = 1, \dots, n-1), \ \eta_j = \partial y_j \ (j = 1, \dots, k), \ \eta_{k+j} = \partial z_j \ (j = 1, \dots, l).$$

Then we see that $\langle \eta_{k+1}, \dots, \eta_{k+l} \rangle_{\mathbf{R}} = \ker \mathcal{H}_\eta$ on $S(f)$. Set

$$\lambda_j = \det(\xi_1 f, \dots, \xi_{n-1} f, \eta_j f), \quad j = 1, \dots, m-n+1, \quad \Lambda = (\lambda_1, \dots, \lambda_{m-n+1}).$$

It follows that

$$\Lambda = (2e_1 y_1, \dots, 2e_k y_k, h_{z_1}(x, z), \dots, h_{z_l}(x, z))$$

and $S(f) = \{\Lambda = 0\}$. By non-degeneracy, we have $d\lambda_0 \neq 0$. The matrix which represents $d\lambda_0$ is given by

$$\begin{aligned} A &= \left(\begin{array}{c|c|c} \left((f_n)_{y_i x_j} \right)_{\substack{i=1, \dots, k, \\ j=1, \dots, n-1}} & \left((f_n)_{y_i y_j} \right)_{i,j=1, \dots, k} & \left((f_n)_{y_i z_j} \right)_{\substack{i=1, \dots, k, \\ j=1, \dots, l}} \\ \hline \left((f_n)_{z_i x_j} \right)_{\substack{i=1, \dots, l, \\ j=1, \dots, n-1}} & \left((f_n)_{z_i y_j} \right)_{\substack{i=1, \dots, l, \\ j=1, \dots, k}} & \left((f_n)_{z_i z_j} \right)_{i,j=1, \dots, l} \end{array} \right) (0) \\ &= \left(\begin{array}{c|c|c|c} * & * & \text{Hess } q(y) & O \\ \hline \left(h_{z_i x_j} \right)_{\substack{i=1, \dots, l, \\ j=1, \dots, l}} & \left(h_{z_i x_j} \right)_{\substack{i=1, \dots, l, \\ j=l+1, \dots, n-1}} & O & \left(h_{z_i z_j} \right)_{i,j=1, \dots, l} \end{array} \right) (0) \\ &=: \left(\begin{array}{c|c|c|c} * & * & \text{Hess } q(y) & O \\ \hline M_1 & M_2 & O & M_3 \end{array} \right) (0), \end{aligned}$$

where O stands for a zero matrix. Since $M_3(0) = O$, we may assume M_1 is regular by a coordinate change if necessary. By the implicit function theorem, there exist functions

$$x_i(x_{l+1}, \dots, x_{n-1}, z) \quad (i = 1, \dots, l), \quad z = (z_1, \dots, z_l),$$

such that

$$(2.10) \quad \begin{aligned} h_{z_j} \left(x_1(x_{l+1}, \dots, x_{n-1}, z), \dots, x_l(x_{l+1}, \dots, x_{n-1}, z), x_{l+1}, z \right) &= 0 \quad (j = 1, \dots, l), \\ x_{l+1} &= (x_{l+1}, \dots, x_{n-1}). \end{aligned}$$

Then $g := f|_{S(f)}$ is expressed by

$$g(x_{l+1}, z) = f \left(x_1(x_{l+1}, \dots, x_l(x_{l+1}, \dots, x_{n-1}, z), x_{l+1}, 0, z) \right).$$

Hence the transportation matrix which represent $d(f|_{S(f)})$ is given by

$$B = \left(\begin{array}{c|c|c} \begin{array}{ccc} (x_1)_{x_{l+1}} & \cdots & (x_l)_{x_{l+1}} \\ \vdots & \vdots & \vdots \\ (x_1)_{x_{n-1}} & \cdots & (x_l)_{x_{n-1}} \end{array} & I & * \\ \hline \begin{array}{ccc} (x_1)_{z_1} & \cdots & (x_l)_{z_1} \\ \vdots & \vdots & \vdots \\ (x_1)_{z_l} & \cdots & (x_l)_{z_l} \end{array} & O & \begin{array}{c} \sum_{i=1}^l h_{x_i}(x_i)_{z_1} + h_{z_1} \\ \vdots \\ \sum_{i=1}^l h_{x_i}(x_i)_{z_l} + h_{z_l} \end{array} \end{array} \right) =: \left(\begin{array}{c|c|c} * & I & * \\ \hline N_1 & O & v \end{array} \right),$$

where I stands for the identity matrix. Since $\partial z_1, \dots, \partial z_l$ are contained in $\ker df$ on $S(f)$, the derivatives h_{z_1}, \dots, h_{z_l} vanishes on $S(f)$, and we have

$$v = N_1 \begin{pmatrix} h_{x_1} \\ \vdots \\ h_{x_l} \end{pmatrix}.$$

Hence, by elementary row operations B changes to

$$(2.11) \quad \left(\begin{array}{c|c|c} * & I & * \\ \hline N_1 & O & O \end{array} \right).$$

Thus $(x, 0, z) \in S(f|_{S(f)})$ is equivalent to the determinant of $N_1(x, 0, z)$ being zero. Differentiating (2.10), we have

$$N_1^t M_1 = - \begin{pmatrix} h_{z_1 z_1} & \cdots & h_{z_1 z_l} \\ \vdots & \ddots & \vdots \\ h_{z_l z_1} & \cdots & h_{z_l z_l} \end{pmatrix}.$$

Since M_1 is regular, $(x, 0, z) \in S(f|_{S(f)})$ is equivalent to $\det \text{Hess } h(0, z) = 0$. On the other hand, $\eta_j \lambda_i = h_{z_i z_j}$ holds on $S(f)$, and we have $\text{Hess } h(0, z) = \mathcal{H}_\eta$. Since $\ker dg = \langle \partial z_1, \dots, \partial z_l \rangle_{\mathbf{R}}$ by (2.11), one can easily see that the last assertion holds true. \square

Set

$$H = \det \mathcal{H}_\eta.$$

Definition 2.7. A non-degenerate singular point 0 is called *2-singular* if $H(0) = 0$.

This is equivalent to $\ker df_0 \cap T_0 S(f) \neq \emptyset$. Set $S_2(f) = \{H = 0\}$. The 2-singularity of a non-degenerate singular point does not depend on the choice of η . By Lemma 2.6, it follows that $S_2(f) = S(g)$.

Definition 2.8. A 2-singular point 0 is *2-non-degenerate* if $d(H|_{S(f)})_0 \neq 0$.

The condition is equivalent to $\ker dH_0 \not\supset T_0 S(f)$. By the definition, we see that the 2-non-degeneracy condition does not depend on the choice of η , and if p is 2-non-degenerate, then $S_2(f)$ is a manifold near p . Moreover, $\text{rank } \mathcal{H}_\eta(0) = m - n$. In fact, if we assume that $\text{rank } \mathcal{H}_\eta(0) < m - n$, then all the minor $m - n - 1$ determinants of $\mathcal{H}_\eta(0)$ vanish. Since dH_0 is expressed by these minor determinants, we have $dH_0 = 0$.

Let p be a 2-singular point. Since $H(p) = 0$, dimension of $\ker \mathcal{H}_\eta(p)$ is positive. Let θ_p be a non-zero element of $\ker \mathcal{H}_\eta(p)$.

Lemma 2.9. If $\text{rank } \mathcal{H}_\eta = m - n$, then there exists a vector field θ on $(\mathbf{R}^m, 0)$ such that θ_p generates $\ker \mathcal{H}_\eta(p)$ when $p \in S_2(f) (= \{H = 0\})$. Namely, $\langle \theta_p \rangle_{\mathbf{R}} = \ker \mathcal{H}_\eta(p)$.

Proof. The matrix \mathcal{H}_η is symmetric on $S(f)$, and has only one zero-eigenvalue at 0. Thus the eigenvalue κ , that has minimum absolute value, is well-defined on a neighborhood U of 0, and it takes a real value on U . We denote that by θ the non-zero eigenvector with respect to κ . Then θ is an eigenvector of the zero eigenvalue on $S_2(f)$ and so, one can extend θ on $(\mathbf{R}^m, 0)$, and get the desired vector field. \square

We state a condition that θ is in the kernel of \mathcal{H}_η .

Lemma 2.10. For $p \in S(f)$, the condition $\theta_p \in \ker \mathcal{H}_\eta(p)$ is equivalent to $\theta \lambda_1 = \dots = \theta \lambda_{m-n+1} = 0$ at p .

Proof. Let $\eta_1, \dots, \eta_{m-n+1}$ be vector fields generating $\ker df$, and set $\theta = \sum_{i=1}^{m-n+1} \theta_i \eta_i$. Then by (2.8) and symmetry of \mathcal{H}_η , we see that

$$\mathcal{H}_\eta(\theta) = \sum_{i=1}^{m-n+1} \left(\sum_{j=1}^{m-n+1} \theta_j \eta_i \lambda_j \right) \eta_i = \sum_{i=1}^{m-n+1} \left(\sum_{j=1}^{m-n+1} \theta_j \eta_j \lambda_i \right) \eta_i = \sum_{i=1}^{m-n+1} (\theta \lambda_i) \eta_i.$$

Thus the assertion holds. \square

If p is a 2-non-degenerate singular point, then $S_2(f)$ is a manifold near p . Thus the condition that θ is tangent to $S_2(f)$ at a point on $S_2(f)$ is well-defined. Hence we introduce the definition below. In what follows, we denote by $'$ the directional derivative along the direction θ . Namely, $H' = \theta H$. Furthermore, $H^{(i)} = (H^{(i-1)})'$ ($i = 2, 3, \dots$) and $H^{(1)} = H'$, $H^{(0)} = H$.

Definition 2.11. A 2-non-degenerate singular point 0 is called 3-singular if $\theta(0) \in T_0 S_2(f)$.

Since the 3-singularity is determined by θ at p , it does not depend on the extension of θ , and $S_2(f)$ does not depend on the extension of η , so the 3-singularity does not depend on the extension of η . We remark that the 3-singularity is equivalent to $H'(0) = 0$. Let us set $S_3(f) = \{q \mid \theta_q \in T_q S_2(f)\}$. Then $S_3(f)$ is determined by θ on $S_2(f)$. Thus $S_3(f)$ does not depend on the extension of η, θ . Furthermore, we see that

$$S_3(f) = \{p \in S_2(f) \mid H'(p) = 0\} = \{p \in S(f) \mid H(p) = H'(p) = 0\}.$$

Using this terminology, 3-singularity is equivalent to $0 \in S_3(f)$. Moreover, we have:

Lemma 2.12. *It holds that $S_3(f) = S(f|_{S_2(f)})$.*

Proof. If $p \in S_2(f)$, it holds that $\ker d(f|_{S_2(f)})_p = \langle \theta_p \rangle_{\mathbf{R}}$. Thus we obtain the result. \square

Definition 2.13. A 3-singular point 0 is 3-non-degenerate if $d(H'|_{S_2(f)})_0 \neq 0$ holds.

Lemma 2.14. *The 3-non-degeneracy condition on a 3-singular point does not depend on the extension of η , on the extension of θ , nor on the coordinate system on the target.*

Proof. Let $\tilde{\theta}$ be another extension of θ . Then $\tilde{\theta}H|_{S_2(f)} = \theta H|_{S_2(f)}$ holds on $S_2(f)$, since the 3-non-degeneracy depends only on the first differential by θ . Thus the 3-non-degeneracy does not depend on the extension of θ . On the other hand, let $\tilde{\eta}$ be another extension of η , and set $\tilde{H} = \det \mathcal{H}_{\tilde{\eta}}$. Then we have $\tilde{H} = \alpha H + \beta$, where $\alpha|_{S(f)} \neq 0$ and $\beta|_{S(f)} = 0$. Thus it holds that $\tilde{H}' = \alpha' H + \alpha H' + \beta'$. We restrict this formula to $S_2(f)$. We see that $\beta' = 0$ on $S_2(f)$, because $H = 0$ and $p \in S_2(f)$ then $\theta_p \in T_p S(f)$ holds. Thus

$$\tilde{H}'|_{S_2(f)} = \alpha H'|_{S_2(f)}$$

holds. On the other hand, if 0 is 3-singular, then by $H'(0) = 0$, we see $d(\tilde{H}'|_{S_2(f)})_0 = \alpha d(H'|_{S_2(f)})_0$. Thus it does not depend on the extension of η . \square

The 3-non-degeneracy is equivalent to $\ker d(H')_0 \not\supset T_0 S_2(f)$. Thus if 0 is 3-non-degenerate, then $S_3(f)$ is a manifold.

Lemma 2.15. *Let 0 be a non-degenerate singular point. Then 0 is 3-non-degenerate if and only if $H = H' = 0$ at 0 and $\text{rank } d(H, H')_0|_{T_0 S(f)} = 2$.*

Proof. Since both conditions imply the 2-non-degeneracy, we assume 0 is 2-non-degenerate. Since 0 is non-degenerate, we take a coordinate systems on the source and target such that $f(x, y, z) = (x, f_n(x, y, z))$ has the form (2.9), and $(f_n)_{z_1 x_1}(0) \neq 0$. Moreover $dH_0 \neq 0$, we see $l = 1$. Then we take an adapted collection of vector fields

$$\begin{aligned}\xi_1 &= \partial x_1, \quad \xi_i = -(f_n)_{z_1 x_i} \partial x_1 + (f_n)_{z_1 x_1} \partial x_i, \quad (i = 2, \dots, n-1), \\ \eta_j &= \partial y_j \quad (j = 1, \dots, m-n), \quad \eta_{m-n+1} = \partial z_1.\end{aligned}$$

Then we see that

$$T_0 S(f) = \langle \partial x_2, \dots, \partial x_{n-1}, \partial z_1 \rangle_{\mathbf{R}}, \quad \partial z_1 \in T_0 S_2(f).$$

We assume that 0 is 3-non-degenerate. Then $H = H' = 0$ at 0, and $dH'_0|_{T_0 S_2(f)} \neq 0$ holds. Thus there exists a vector $\xi \in T_0 S_2(f)$ such that $\xi H'(0) \neq 0$. Since $S_2(f) = \{H = 0\}$, it holds that $\xi H = 0$. By $dH_0|_{T_0 S(f)} \neq 0$, it holds that $\text{rank } d(H, H')_0|_{T_0 S(f)} = 2$. On the other hand, $\xi H = 0$ holds for $\xi \in T_0 S_2(f)$. Hence we see that $\text{rank } d(H, H')_0|_{T_0 S(f)} = 2$ implies $d(H')_0|_{S_2(f)} \neq 0$. \square

We define $(i+1)$ -singularity and $(i+1)$ -non-degeneracy inductively. Let the notion of j -singularity, the set of j -singular points $S_j(f) = \{p \in (\mathbf{R}^m, 0) \mid H(p) = \dots = H^{(i-2)}(p) = 0\}$ as a manifold, and j -non-degeneracy already be defined for $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ ($j = 1, \dots, i$). Moreover, we assume that these notions do not depend on the extensions of η and θ . Here 1-non-degenerate means non-degenerate, and 1-singular point means singular point.

Definition 2.16. An i -non-degenerate singular point 0 is $(i+1)$ -singular if $\theta \in T_0 S_i(f)$.

We remark that, since the $(i+1)$ -singularity is defined only by the condition of θ be on $S_i(f)$ and $S_i(f)$ itself, then it does not depend on the extension of η and θ . We set $S_{i+1}(f) = \{\theta_p \in T_p S_i(f)\}$. Then $S_{i+1}(f)$ also does not depend on the extension of η and θ , and we have

$$S_{i+1}(f) = \{p \in (\mathbf{R}^m, 0) \mid H(p) = \dots = H^{(i-1)}(p) = 0\}.$$

Definition 2.17. An $(i+1)$ -singular point 0 is $(i+1)$ -non-degenerate if $d(H^{(i-1)})|_{S_i(f)}|_0 \neq 0$ holds.

Lemma 2.18. The $(i+1)$ -non-degeneracy does not depend on the extensions of θ and η .

Proof. We show this for the extension of θ . Let $\tilde{\theta}$ be a vector field satisfying that $\tilde{\theta}|_{S_2(f)} = \delta\theta|_{S_2(f)}$ ($\delta \neq 0$). It is enough to show that $\delta^{i-1}H^{(i-1)}|_{S_i(f)} = \tilde{\theta}^{i-1}H|_{S_i(f)}$. We show it by induction. We set $\tilde{\theta} = \delta\theta + \gamma$, where γ is a vector field which satisfies $\gamma|_{S_2(f)} = 0$. When $i = 2$, we see the conclusion. We assume that $(H^{(i-2)} - \delta^{i-2}\tilde{\theta}^{i-2}H)|_{S_{i-1}(f)} = 0$. Then by

$$\tilde{\theta}^{i-1}H|_{S_{i-1}(f)} = \tilde{\theta}\tilde{\theta}^{i-2}H|_{S_{i-1}(f)} = (\delta\theta + \gamma)\tilde{\theta}^{i-2}H|_{S_{i-1}(f)} = \delta\theta\tilde{\theta}^{i-2}H|_{S_{i-1}(f)},$$

we see that

$$\begin{aligned}& (\delta^{i-1}H^{(i-1)} - \tilde{\theta}^{i-1}H)|_{S_{i-1}(f)} \\ &= \delta\left(\delta^{i-2}H^{(i-2)} - \tilde{\theta}^{i-2}H\right)|_{S_{i-1}(f)} \\ &= \left(\delta\theta(\delta^{i-2}H^{(i-2)} - \tilde{\theta}^{i-2}H) - (\theta\delta^{i-2}H^{(i-2)})\right)|_{S_{i-1}(f)}.\end{aligned}$$

By the assumption of induction, $(\delta^{i-2}H^{(i-2)} - \tilde{\theta}^{i-2}H)|_{S_{i-1}(f)} = 0$ holds. Since $\theta \in TS_{i-1}(f)$ and $\theta^{i-2}H = 0$ hold on $S_i(f)$, we see that

$$\left(\theta(\delta^{i-2}H^{(i-2)} - \tilde{\theta}^{i-2}H)\right)|_{S_{i-1}(f)} = (\theta\delta^{i-2}H^{(i-2)})|_{S_{i-1}(f)} = 0.$$

(2) We take another extension $\tilde{\eta}$ of η , and let $\mathcal{H}_{\tilde{\eta}} = \tilde{H}$. Then by Lemma 2.5 we see that $\tilde{H} = \alpha H + \beta$ holds, where $\alpha|_{S(f)} \neq 0$ and $\beta|_{S(f)} = 0$. Then by the same method as in the proof of Lemma 2.14, one can see $(\alpha H^{(i-1)} - \tilde{H}^{(i-1)})|_{S_i(f)} = 0$, by using $S_i(f) = \{p \in S_{i-1}(f) \mid H^{(i-2)}(p) = 0\}$, which proves the $(i+1)$ -non-degeneracy does not depend on the extension of η . \square

We remark that $(i+1)$ -non-degeneracy is equivalent to $\ker d(H^{(i-1)})_0 \not\supset T_0 S_i$, and we can continue until $S_i(f)$ becomes a point, namely $i = n$. Since $T_0 S_n = \{0\}$, the $(n+1)$ -singularity always fails. On other words, n -non-degeneracy implies $(n+1)$ -non-degeneracy and so on. In fact, by the definition, n -non-degeneracy implies $d(H^{(n-2)})|_{S_{n-1}(f)}_0 \neq 0$. Since S_{n-1} is one-dimensional, if $\theta_p \in T_p S_{n-1}(f)$, then $\langle \theta_p \rangle_{\mathbf{R}} = T_p S_{n-1}(f)$ holds, and $\theta(H^{(n-2)}|_{S_{n-1}(f)})(0) \neq 0$ follows.

Lemma 2.19. *Let us assume that $i \leq n$, and 0 is a non-degenerate singular point. Then the i -non-degeneracy is equivalent to $H(0) = H'(0) = \dots = H^{(i-2)}(0) = 0$ and $\text{rank } d(H, H', \dots, H^{(i-2)})_0|_{T_0 S(f)} = i - 1$.*

Proof. By induction we assume that the conclusion is true for $1, \dots, i-1$, and that 0 is an i -non-degenerate singular point. Then we have $H(0) = H'(0) = \dots = H^{(i-2)}(0) = 0$ and $\text{rank } d(H, H', \dots, H^{(i-3)})_0 = i - 2$.

We take a coordinate system (x_1, \dots, x_{n-2}, z) of S satisfying

$$\text{rank} \begin{pmatrix} H_{x_1} & \cdots & H_{x_{i-2}} \\ H'_{x_1} & \cdots & H'_{x_{i-2}} \\ \vdots & \vdots & \vdots \\ H_{x_1}^{(i-3)} & \cdots & H_{x_{i-2}}^{(i-3)} \end{pmatrix} (0) = i - 2$$

and $\theta = \partial z$. The transposition of the matrix representation of $d(H, H', \dots, H^{(i-2)})_0$ with respect to this coordinate system is

$$\left(\begin{array}{c|c} K_1 & L_1 \\ \hline K_2 & L_2 \end{array} \right) := \left(\begin{array}{ccc|c} H_{x_1} & \cdots & H_{x_1}^{(i-3)} & H_{x_1}^{(i-2)} \\ \vdots & \vdots & \vdots & \vdots \\ H_{x_{i-2}} & \cdots & H_{x_{i-2}}^{(i-3)} & H_{x_{i-2}}^{(i-2)} \\ \hline H_{x_{i-1}} & \cdots & H_{x_{i-1}}^{(i-3)} & H_{x_{i-1}}^{(i-2)} \\ \vdots & \vdots & \vdots & \vdots \\ H_{x_{n-2}} & \cdots & H_{x_{n-2}}^{(i-3)} & H_{x_{n-2}}^{(i-2)} \\ H' & \cdots & H^{(i-2)} & H^{(i-1)} \end{array} \right).$$

By elementary matrix operations, the above matrix is deformed to

$$\left(\begin{array}{c|c} K_1 & O \\ \hline K_2 & L_2 - K_2 K_1^{-1} L_1 \end{array} \right).$$

Now applying the implicit function theorem, we see that

$$X := \begin{pmatrix} (x_1)_{x_{i+1}} & \cdots & (x_i)_{x_{i+1}} \\ \vdots & \vdots & \vdots \\ (x_1)_{x_{n-2}} & \cdots & (x_i)_{x_{n-2}} \\ (x_1)' & \cdots & (x_i)' \end{pmatrix} = -K_2 K_1^{-1}.$$

Since

$$\begin{pmatrix} (H^{(i-2)}|_{S_i})_{x_{i+1}} \\ \vdots \\ (H^{(i-2)}|_{S_i})_{x_{n-2}} \\ (H^{(i-2)}|_{S_i})' \end{pmatrix} = {}^t(XL_1 + L_2) = {}^t(-K_2K_1^{-1}L_1 + L_2),$$

we have the conclusion.

For the converse, if we assume that $H(0) = H'(0) = \dots = H^{(i-2)}(0) = 0$ and $\text{rank } d(H, H', \dots, H^{(i-3)})_0 = i - 2$, we can see the conclusion just following the arguments above from the bottom up. \square

3 Criteria

Theorem 3.1. *A map-germ $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ is \mathcal{A} -equivalent to the k -Morin singularity ($k = 1, \dots, n$) if and only if 0 is a k -non-degenerate singularity but not $(k+1)$ -singular.*

Proof. Since the k -non-degenerate conditions and $(k+1)$ -singularity conditions do not depend on the coordinate systems on the source nor the target space, the sufficiency is obvious by just checking the normal form (1.1) of the Morin singularity. We now show the necessity. Let us assume that 0 is a k -non-degenerate singularity but not $(k+1)$ -singular. Since the assumption does not depend on the coordinate systems on the source and the target space, and since $\text{rank } df_0 = n - 1$, we take a coordinate system such that (2.2) holds. When $k = 1$, we see that $\text{Hess } h(0, y)$ at $y = 0$ is regular. By the parametrized Morse lemma, we have the conclusion. We now assume $2 \leq k \leq n$. It is known that the assumption does not depend on the extension of θ , and by Lemma 2.5 and (2.7), H_f is multiplied by a non-zero function when changing η . Moreover, the condition also does not depend on the extension of η , hence we may take $\eta_i = \partial y_i$, and coordinate z such that $\theta = \partial z$ at 0. Under this coordinate system, we rewrite f as

$$f(x, y, z) = (x, h(x, y, z)), \quad x = (x_1, \dots, x_{n-1}), \quad y = (y_1, \dots, y_{m-n}).$$

Then $\text{Hess } h(0, y, 0)$ is regular at $y = 0$, by the parametrized Morse lemma, we may choose coordinates y such that f takes the form

$$f(x, y, z) = (x, q(y) + h(x, z)), \quad q(y) = \sum_{i=1}^{m-n} \pm y_i^2, \\ x = (x_1, \dots, x_{n-1}), \quad y = (y_1, \dots, y_{m-n}).$$

Then we see that $\theta = \partial z$. Under this coordinate system, $H(0) = \theta^2 h(0)$ and $\theta H(0) = \theta^3 h(0)$ hold. Moreover, the i -non-degeneracy implies that

$$(\theta h_{x_1}, \dots, \theta h_{x_{n-1}})(0) \neq (0, \dots, 0).$$

We set

$$\bar{f}(x, z) = f(x, 0, z) = (x, g(x, z)) : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0).$$

Then $\bar{f}(x, z)$ at 0 is an A_k -Morin singularity in the sense of [25]. Because the Jacobian is $\lambda := \det J_{\bar{f}} = g_z$, and $\eta := \partial z$ generates the kernel of $d\bar{f}$ at the singular set. Thus by the assumption of i -non-degeneracy, we have $\lambda = \eta\lambda = \dots = \eta^{k-1}\lambda = 0$, $\eta^k\lambda \neq 0$, $\text{rank } d(\lambda, \eta\lambda, \dots, \eta^{k-1}\lambda)_0 = k$. Hence, it follows by ([25, Theorem A1]), that $\bar{f}(x, z)$ is \mathcal{A} -equivalent to $(x, z^{k+1} + \sum_{i=1}^{k-1} x_i z^i)$. Since $\det \text{Hess } q(0) \neq 0$, we see the assertion. \square

The proof here is based on that of Morin [17]. By Lemma 2.19, we have the following:

Theorem 3.2. Let 0 be a non-degenerate singular point of $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$. Then f at 0 is a k -Morin singularity ($2 \leq k \leq n$) if and only if both conditions above holds true:

$$(1) \ H = H' = \dots = H^{(k-2)} = 0, \ H^{(k-1)} \neq 0,$$

$$(2) \ \text{rank } d(H, H', \dots, H^{(k-2)})_0|_{T_0S(f)} = k - 1.$$

Here, \mathcal{H}_η is determined by (2.5) for an adapted collection of vector fields (ξ, η) with respect to f , $H = \det \mathcal{H}_\eta$, and θ is a vector field such that it generates $\ker \mathcal{H}_\eta$ on $\{H = 0\}$, and $'$ means the directional derivative along θ .

Moreover we have the following corollary.

Corollary 3.3. Let 0 be a singular point of $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ satisfying $\text{rank } df_0 = n - 1$. Then f is a k -Morin singularity ($2 \leq k \leq n$) at 0 if and only if

$$(a) \ H = H' = \dots = H^{(k-2)} = 0, \ H^{(k-1)} \neq 0,$$

$$(b) \ \text{rank } d(\lambda_1, \dots, \lambda_{m-n+1}, H, H', \dots, H^{(k-2)})_0 = m - n + k,$$

where H is the same as in Theorem 3.2.

Proof. Since the condition (a) is the same, we show that (b) is equivalent to non-degeneracy and the condition (2) of Theorem 3.2. The condition does not depend on the choice of an adapted collection of vector fields, so we choose $\xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_{m-n}, \theta$ satisfying $\xi_2, \dots, \xi_{n-1} \in T_0S(f)$, since $\langle \eta_1, \dots, \eta_{m-n} \rangle_{\mathbf{R}} \cap T_0S(f) = \{0\}$. Then the transportation of the matrix representation of the differential $d(\lambda_1, \dots, \lambda_{m-n+1}, H, H', \dots, H^{(k-2)})_0$ has the form

$$= \begin{pmatrix} \begin{array}{c|c|c|c} \xi_1 \lambda_1 & (\xi_j \lambda_i)_{\substack{i=1, \dots, m-n, \\ j=2, \dots, n-1}} & (\eta_j \lambda_i)_{\substack{i=1, \dots, m-n, \\ j=1, \dots, m-n}} & \lambda'_1 \\ \vdots & & & \vdots \\ \xi_1 \lambda_{m-n} & & & \lambda'_{m-n} \end{array} \\ \hline \begin{array}{c|c|c|c} \xi_1 \lambda_{m-n+1} & (\xi_j \lambda_{m-n+1})_{j=2, \dots, n-1} & (\eta_j \lambda_{m-n+1})_{j=1, \dots, m-n} & \lambda'_{m-n+1} \end{array} \\ \hline \begin{array}{c|c|c|c} \xi_1 H & (\xi_j H^{(i)})_{\substack{i=0, \dots, k-3, \\ j=2, \dots, n-1}} & (\eta_j H^{(i)})_{\substack{i=0, \dots, k-3, \\ j=1, \dots, m-n}} & H' \\ \vdots & & & \vdots \\ \xi_1 H^{(k-3)} & & & H^{(k-2)} \end{array} \\ \hline \begin{array}{c|c|c|c} \xi_1 H^{(k-2)} & (\xi_j H^{(k-2)})_{j=2, \dots, n-1} & (\eta_j H^{(k-2)})_{j=1, \dots, m-n} & H^{(k-1)} \end{array} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{array}{c|c|c|c} * & & & 0 \\ \vdots & & & \vdots \\ * & & & 0 \end{array} \\ \hline \begin{array}{c|c|c|c} \xi_1 \lambda_{m-n+1} & 0 & \dots & 0 \\ \hline * & & & 0 \\ \vdots & (\xi_j H^{(i)})_{\substack{i=0, \dots, k-3, \\ j=2, \dots, n-1}} & * & \vdots \\ * & & & 0 \end{array} \\ \hline \begin{array}{c|c|c|c} * & * & * & H^{(k-1)} \end{array} \end{pmatrix}$$

at the origin. Thus we see that the condition (b) is equivalent to $\xi_1 \lambda_{m-n+1}(0) \neq 0$ and

$$\text{rank} \begin{pmatrix} \xi_2 H & \dots & \xi_{n-1} H \\ \vdots & \vdots & \vdots \\ \xi_2 H^{(k-3)} & \dots & \xi_{n-1} H^{(k-3)} \end{pmatrix} (0) = k - 2.$$

This is nothing but the non-degeneracy and the condition (2) of Theorem 3.2. \square

4 Criteria for small k

In this section, we remark that for small k , the criteria can be simplified. In what follows, for real numbers $a, b \in \mathbf{R}$, the notation $a \sim b$ implies $a = 0$ is equivalent to $b = 0$, and for functions f, g , the notation $f \sim g$ implies that g is multiplication by a non-zero function f .

4.1 Criterion of the fold

Corollary 4.1. *Let $f = (f_1, \dots, f_n) : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ be a map-germ satisfying $(df_n)_0 = 0$ and $\text{rank } df_0 = n - 1$. Then f is a fold singularity at 0 if and only if $\text{rank } \text{Hess}_{\hat{\eta}} f_n = m - n + 1$, where $\hat{\eta}_1, \dots, \hat{\eta}_{m-n+1}$ are vector fields satisfying that $\langle \hat{\eta}_1, \dots, \hat{\eta}_{m-n+1} \rangle_{\mathbf{R}} = \ker df$ at 0. Here, the number of minus signs in q is equal to the number of negative eigenvalues of $\text{Hess}_{\hat{\eta}} f_n$.*

Proof. Let $\xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_{m-n+1}$ be an adapted collection of vector fields with respect to f . Then f is a fold singularity at 0 if and only if non-degeneracy holds with $H(0) \neq 0$. Since $H(0) \neq 0$ implies that $\text{rank } \mathcal{H}_\eta = m - n + 1$, and $d\Lambda_0$ contains \mathcal{H}_η , the non-degeneracy follows from $H(0) \neq 0$. Thus f is the fold if and only if $H(0) \neq 0$. On the other hand, by $(df_n)_0 = 0$, we see $\eta_j \lambda_i(0) = \delta_{\eta_j} \eta_i f_n(0)$, where

$$\delta = \det \begin{pmatrix} \xi_1 f_1 & \cdots & \xi_{n-1} f_1 \\ \vdots & \vdots & \vdots \\ \xi_1 f_{n-1} & \cdots & \xi_{n-1} f_{n-1} \end{pmatrix}.$$

Thus $H = \det \text{Hess}_\eta f_n(0)$. Furthermore, since $\eta_1, \dots, \eta_{m-n+1}$ satisfies $\eta_i f_n = 0$, we see that $\eta_i(0) = \hat{\eta}_i(0)$ implies $\det \text{Hess}_{\hat{\eta}} f_n(0) = \det \text{Hess}_\eta f_n(0)$. \square

4.2 Criterion of the cusp

For a function-germ $t : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}, 0)$ which has a critical point at 0 and a subspace $V \subset T_0 \mathbf{R}^n$, we consider the Hessian matrix $(v_j v_i t)_{(1 \leq i, j \leq k)}$ with respect to a basis v_1, \dots, v_k of V , which is defined by $(\tilde{v}_j \tilde{v}_i t)_{(1 \leq i, j \leq k)}(0)$, where \tilde{v}_i is an extension of v_i . We remark that since t has a critical point at 0, it does not depend on the choice of extensions. Moreover, the $\ker(v_j v_i t)_{(1 \leq i, j \leq k)}$ depends only on V . We denote it by $\ker \text{Hess}_V h(0)$.

Corollary 4.2. *Let $f = (f_1, \dots, f_n) : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$ be a map-germ satisfying $(df_n)_0 = 0$ and $\text{rank } df_0 = n - 1$. Then f is a cusp singularity at 0, if and only if for a vector field $\hat{\theta}$ satisfying*

$$\ker \text{Hess}_{\ker df_0} f_n(0) = \langle \hat{\theta}_0 \rangle_{\mathbf{R}},$$

and contained in the $\ker df$ on $S(f)$, it holds that

$$(1) \quad \hat{\theta}^3 f_n(0) \neq 0,$$

$$(2) \quad d(\hat{\theta} f_n)_0 \neq 0.$$

Here, $\text{rank } \text{Hess}_{\ker df_0} f_n(0) = m - n$ and the number of negative eigenvalues is equal to the number of minus signs in q .

Proof. The necessity is obvious, we show the sufficiency. By Theorem 3.2, we show non-degeneracy, 2-non-degeneracy and non-3-singularity. Namely, we show that the conditions (1) and (2) imply non-degeneracy, and $H(0) = 0$, $H'(0) \neq 0$ and $\text{rank } dH_0|_{T_0 S(f)} = 1$. If θ is a generator of the kernel of \mathcal{H}_f , then since 0 is 2-singular i.e., $\theta_0 \in T_0 S(f)$, $\text{rank } dH_0|_{T_0 S(f)} = 1$ follows by $H'(0) \neq 0$. Thus it is enough to show non-degeneracy, $H(0) = 0$ and $H'(0) \neq 0$.

Before showing that, we give some calculations. Since the conditions do not depend on the choice of η , we take an adapted collection $(\xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_{m-n}, \theta)$ of vector fields with respect to f . Since θ belongs to the kernel of \mathcal{H}_η on $S_2(f)$, and $0 \in S_2(f)$, so it holds that

$$\theta \lambda_i = 0 \quad (i = 1, \dots, m-n+1) \quad \text{at } 0.$$

On the other hand, $\eta_i f = 0$ ($i = 1, \dots, m-n$) and $\theta f = 0$ hold on $S(f)$, so it holds that

$$\theta \eta_i f = 0 \quad (i = 1, \dots, m-n) \quad \text{and} \quad \theta^2 f = 0 \quad \text{at } 0.$$

Moreover, by $(df_n)_0 = 0$ and $[\theta, \eta_i] \in T\mathbf{R}^m$, it holds that $\eta_i \theta f_n = 0$ at 0. Thus the bottom column of

$$\eta_i \lambda_{m-n+1}(0) = \det(\xi_1 f, \dots, \xi_{n-1} f, \eta_i \theta f)(0)$$

is 0, and we see that $\eta_i \lambda_{m-n+1}(0) = 0$ ($i = 1, \dots, m-n$). We remark that the kernel of $\text{Hess}_\eta f_n(0)$ is θ_0 .

We translate the conditions $H = 0$ and $H' \neq 0$ using f_n . By the above calculation, it follows that

$$\begin{aligned} \theta H(0) &= \theta \det \left(\begin{array}{ccc|c} \eta_1 \lambda_1 & \cdots & \eta_{m-n} \lambda_1 & \theta \lambda_1 \\ \vdots & \vdots & \vdots & \vdots \\ \eta_1 \lambda_{m-n} & \cdots & \eta_{m-n} \lambda_{m-n} & \theta \lambda_{m-n} \\ \hline \eta_1 \lambda_{m-n+1} & \cdots & \eta_{m-n} \lambda_{m-n+1} & \theta \lambda_{m-n+1} \end{array} \right) (0) \\ &= \det \left(\begin{array}{ccc|c} \eta_1 \lambda_1 & \cdots & \eta_{m-n} \lambda_1 & \theta^2 \lambda_1 \\ \vdots & \vdots & \vdots & \vdots \\ \eta_1 \lambda_{m-n} & \cdots & \eta_{m-n} \lambda_{m-n} & \theta^2 \lambda_{m-n} \\ \hline \eta_1 \lambda_{m-n+1} & \cdots & \eta_{m-n} \lambda_{m-n+1} & \theta^2 \lambda_{m-n+1} \end{array} \right) (0) \\ &\sim \theta^2 \lambda_{m-n+1}(0) \\ &= \theta^2 \det(\xi_1 f, \dots, \xi_{n-1} f, \theta f)(0) \\ &= \det(\xi_1 f, \dots, \xi_{n-1} f, \theta^3 f)(0) \\ &\sim \theta^3 f_n(0). \end{aligned}$$

By the same calculation, $H(0) = \theta^2 f_n(0)$ also holds. Thus it is necessary to show that $\theta^2 f_n(0) \sim \hat{\theta}^2 f_n(0)$ and $\theta^3 f_n(0) \sim \hat{\theta}^3 f_n(0)$. We set

$$\hat{\theta} = \sum_{i=1}^{m-n} \alpha_i \eta_i + \beta \theta, \quad \alpha_1(0) = \cdots = \alpha_{m-n}(0) = 0, \quad \beta(0) \neq 0.$$

Then we have

$$\begin{aligned} \hat{\theta} f_n &= \sum_{i=1}^{m-n} \alpha_i \eta_i f_n + \beta \theta f_n, \\ \hat{\theta}^2 f_n &= \sum_{i,j=1}^{m-n} \underline{\alpha_j} (\eta_j \alpha_i \underline{\eta_i f_n} + \underline{\alpha_i} \eta_j \eta_i f_n) + \sum_{j=1}^{m-n} \underline{\alpha_j} (\eta_j \beta \underline{\theta f_n} + \beta \underline{\eta_j \theta f_n}) \\ &\quad + \beta \left(\sum_{i=1}^{m-n} (\theta \alpha_i \underline{\underline{\eta_i f_n}} + \underline{\alpha_i} \theta \underline{\eta_i f_n}) + \theta \beta \underline{\underline{\theta f_n}} + \beta \underline{\theta^2 f_n} \right). \end{aligned}$$

Here, we underline the terms that vanish at the origin, and we put double underlines under the terms that vanish at the origin and whose differentiation along θ vanishes at the origin. Thus we have

$$\hat{\theta}^3 f_n(0) = \theta \hat{\theta}^2 f_n(0) \sim \theta^3 f_n(0).$$

By the same reason, we have $\hat{\theta}^2 f_n(0) \sim \theta^2 f_n(0)$.

Now we show the non-degeneracy condition. We have

$$\begin{aligned} d\Lambda_0 &= \left(\begin{array}{c|c|c} (\xi_j \lambda_i)_{\substack{i=1,\dots,m-n, \\ j=1,\dots,n-1}} & (\eta_j \lambda_i)_{i,j=1,\dots,m-n} & \begin{array}{c} \lambda'_1 \\ \vdots \\ \lambda'_{m-n} \end{array} \\ \hline (\xi_j \lambda_{m-n+1})_{j=1,\dots,n-1} & (\eta_j \lambda_{m-n+1})_{j=1,\dots,m-n} & \lambda'_{m-n+1} \end{array} \right) (0) \\ &= \left(\begin{array}{ccc|ccc} * & \cdots & * & A & 0 & \\ \hline \xi_1 \lambda_{m-n+1} & \cdots & \xi_{n-1} \lambda_{m-n+1} & 0 & \cdots & 0 \end{array} \right) (0). \end{aligned}$$

By 2-non-degeneracy, A is regular, and the non-degeneracy is equivalent to

$$(\xi_1 \lambda_{m-n+1}, \dots, \xi_{n-1} \lambda_{m-n+1})(0) \neq (0, \dots, 0).$$

Moreover, by

$$\xi_i \lambda_{m-n+1}(0) = \det(\xi_1 f, \dots, \xi_{n-1} f, \xi_i \theta f)(0) \sim \xi_i \theta f_n(0),$$

$\eta_i \theta f_n(0) = 0$ ($i = 1, \dots, m-n$) and $\theta^2 f_n(0) = 0$, the condition (2) is equivalent to the non-degeneracy. \square

It should be remarked that by Corollary 4.2, one can easily see that $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^2, 0)$ is a cusp singularity at 0 if and only if 0 is non-degenerate and $f|_{S(f)}$ is \mathcal{A} -equivalent to $t \mapsto (t^2, t^3)$. This criteria was also obtained in [18].

5 First degree bifurcation of Lefschetz singularity

The Lefschetz singularity is a map-germ $(\mathbf{R}^4, 0) \rightarrow (\mathbf{R}^2, 0)$ defined by

$$L(x_1, x_2, y_1, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

This is obtained by considering a map-germ $\mathbf{R}^4 = \mathbf{C}^2 \ni (z, w) \mapsto zw \in \mathbf{C} = \mathbf{R}^2$. From the view point of low-dimensional topology there are many studies of bundles on surfaces with this kind of singular points called the Lefschetz fibrations (See [3, 8], for example.). The Lefschetz singularity is not a stable germ, and it is natural to consider stable perturbations of it. The wrinkling

$$L_w(x_1, x_2, y_1, y_2, s) = (x_1 x_2 - y_1 y_2 + s(x_1 + x_2), x_1 y_2 + x_2 y_1)$$

due to Lekili [15] is such a move and has been well studied. The Lefschetz singularity is not finitely \mathcal{A} -determined, and one cannot obtain a kind of bifurcation diagram. Let us consider

$$\begin{aligned} \tilde{L}_n(x_1, x_2, y_1, y_2, a_1, a_2, b_1, b_2, c_{2000}, \dots, d_{2000}, \dots) \\ = (x_1 x_2 - y_1 y_2 + a_1 x_1 + a_2 x_2, x_1 y_2 + x_2 y_1 + b_1 x_1 + b_2 x_2) \\ + \left(\sum_{i+j+k+l=2}^n c_{ijkl} x_1^i x_2^j y_1^k y_2^l, \sum_{i+j+k+l=2}^n d_{ijkl} x_1^i x_2^j y_1^k y_2^l \right). \end{aligned}$$

Then it holds that

$$\left(tL(\oplus^4 \mathcal{E}_4) + \sum_{k=1}^4 \mathbf{R} \frac{\partial \tilde{L}_n}{\partial a_k} \Big|_p + \mathbf{R} \frac{\partial \tilde{L}_n}{\partial c_{2000}} \Big|_p + \cdots + \mathbf{R} \frac{\partial \tilde{L}_n}{\partial d_{2000}} \Big|_p + \cdots \right) + \oplus^2 \mathcal{M}_4^{n+1} = \oplus^2 \mathcal{E}_4,$$

where $a_3 = b_1, a_4 = b_2$ and $p = (x_1, x_2, y_1, y_2, 0, \dots, 0)$. Here, \mathcal{E}_4 is the set of function-germs $(\mathbf{R}^4, 0) \rightarrow \mathbf{R}$ and \mathcal{M}_4 is its unique maximal ideal, and $tL : \oplus^4 \mathcal{E}_4 \rightarrow \oplus^2 \mathcal{E}_4$ stands for the tangential map:

$$tL(v_1, v_2, v_3, v_4) = \begin{pmatrix} a_1 + x_2 & a_2 + x_1 & -y_2 & -y_1 \\ b_1 + y_2 & b_2 + y_1 & x_2 & x_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

Thus we would like to say that \tilde{L}_n is a “versal-like” unfolding of L up to n -degrees. See [16] for the definition of the versal unfolding.

In [10], deformations of Brieskorn polynomials which include the Lefschetz singularity is considered, and an evaluation of the number of cusp appearing on it is obtained. Here, we would consider the set

$$N = \left\{ C = (a_1, a_2, b_1, b_2) \mid \text{there exists } q = (x_1, x_2, y_1, y_2) \in S(\tilde{L}_1) \text{ such that } \tilde{L}_1 \text{ at } q \text{ is not the fold nor the cusp.} \right\}.$$

We call N the *non-cusp locus*. Although the bifurcation diagram for L cannot be drawn in any finite dimensional space, we can draw $N|_{b_2=\varepsilon} \subset \mathbf{R}^3$ for small ε , and the author believes that we might regard N as a 1 degree bifurcation diagram of L .

We set $\tilde{L}(x_1, x_2, y_1, y_2) = \tilde{L}_1(x_1, x_2, y_1, y_2, a_1, a_2, b_1, b_2)$, regarding a_1, a_2, b_1, b_2 as constants. To detect N , we consider the following three conditions for $q = (x_1, x_2, y_1, y_2)$:

- (i) $\text{rank } d\tilde{L}_q = 0$,
- (ii) $\text{rank } d\tilde{L}_q = 1$ and $\text{rank } \mathcal{H}_\eta = 0$,
- (iii) $\text{rank } d\tilde{L}_q = 1$, $\text{rank } \mathcal{H}_\eta(q) = 1$ and $H(q) = \theta H(q) = 0$.

See Theorem 3.2 for the notations.

Let $C = (a_1, a_2, b_1, b_2) \in N$ and $q = (x_1, x_2, y_1, y_2) \in S(\tilde{L})$ satisfies the condition (iii). We assume that $a_1 + x_2 \neq 0$, then $\eta_1 = (a_2 + x_1)\partial x_1 + (a_1 + x_2)\partial x_2, \eta_2 = y_2\partial x_1 + (a_1 + x_2)\partial y_1, \eta_3 = y_1\partial x_1 + (a_1 + x_2)\partial y_2$ form a basis of the kernel of df at $p = (x_1, x_2, y_1, y_2) \in S(\tilde{L})$. Moreover, ∂x_1 together with η_1, η_2, η_3 forms a basis of $T_p \mathbf{R}^4$. Then we define $\lambda_i = \det(\tilde{L}_{x_1}, \eta_i \tilde{L})$, $\mathcal{H}_{(\eta_1, \eta_2, \eta_3)} = (\eta_i \lambda_j)_{i,j=1,2,3}$ and $H = \det \mathcal{H}_{(\eta_1, \eta_2, \eta_3)}$. Then we see

$$\begin{aligned} \lambda_1 &= -x_1 y_2 + x_2 y_1 - b_1 x_1 + b_2 x_2 + a_1 y_1 - a_2 y_2 + a_1 b_2 - a_2 b_1, \\ \lambda_2 &= x_2^2 + y_2^2 + a_1 x_2 + b_1 y_2, \\ \lambda_3 &= x_1 x_2 + y_1 y_2 + a_1 x_1 + b_1 y_1. \end{aligned}$$

If $(b_1 + 2y_2, (a_1 + 2x_2)y_1) \neq (0, 0)$, then we set

$$\theta = \det \begin{pmatrix} \eta_2 \lambda_1 & \eta_2 \lambda_2 \\ \eta_3 \lambda_1 & \eta_3 \lambda_2 \end{pmatrix} \eta_1 + \det \begin{pmatrix} \eta_3 \lambda_1 & \eta_3 \lambda_2 \\ \eta_1 \lambda_1 & \eta_1 \lambda_2 \end{pmatrix} \eta_2 + \det \begin{pmatrix} \eta_1 \lambda_1 & \eta_1 \lambda_2 \\ \eta_2 \lambda_1 & \eta_2 \lambda_2 \end{pmatrix} \eta_3.$$

Then θ forms a basis of $\ker \mathcal{H}$ on $S_2(\tilde{L})$. We assume $(x_1, y_1) \neq (0, 0)$. If $y_1 = 0$, then $x_2 = -a_1$ since $\lambda_3 = 0$. Thus we may assume $y_1 \neq 0$. Moreover, we may assume $b_1 + y_2 \neq 0$, because if $b_1 + y_2 = 0$ and $x_1 \neq 0$, then we see $a_1 + x_2 = 0$ by $\lambda_3 = 0$. If $b_1 + y_2 = 0$ and $x_1 = 0$, then we see $x_2 = 0$ by $\lambda_2 = 0$. Then we see $a_1 \neq 0$. Then we have $b_2 + y_1 = 0$ since $\lambda_1 = 0$. In this case, $H = 0$ and $\theta H = 0$ can be calculated as

$$y_1 = -(a_2/a_1)y_2, \quad a_1^2 a_2 y_2 (a_1^2 + y_2^2) = 0.$$

Thus we have $y_1 = 0$ which is a contradiction. By the above discussion, if $(x_1, y_1) \neq (0, 0)$, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$ can be modified to

$$(5.1) \quad x_2 = -\frac{x_1(a_1x_1 + b_1y_1)}{x_1^2 + y_1^2}, \quad y_2 = -\frac{y_1(a_1x_1 + b_1y_1)}{x_1^2 + y_1^2}, \quad a_2x_1 + y_1b_2 + x_1^2 + y_1^2 = 0.$$

Substituting (5.1) into $H = 0$, we have

$$(b_1x_1 - a_1y_1)(a_1y_1(-3x_1^2 + y_1^2 - 2a_2x_1) + b_1(x_1^3 - 3x_1y_1^2 + a_2x_1^2 - a_2y_1^2)) = 0.$$

If we assume $b_1x_1 - a_1y_1 = 0$, then we obtain $a_1 + x_2 = 0$ by (5.1). Thus we may assume $b_1x_1 - a_1y_1 \neq 0$. Moreover, if we assume $(-3x_1^2 + y_1^2 - 2a_2x_1, x_1^3 - 3x_1y_1^2 + a_2x_1^2 - a_2y_1^2) = (0, 0)$, then we obtain $x_1 = y_1 = 0$. Thus we may assume $(-3x_1^2 + y_1^2 - 2a_2x_1, x_1^3 - 3x_1y_1^2 + a_2x_1^2 - a_2y_1^2) \neq (0, 0)$. Then we see that $\theta H = 0$ on $H = 0$ is equivalent to $(a_2 + x_1)(a_2 + 2x_1) = 0$. Hence we have a part of N :

$$N_1 = \{(a_1, a_2, b_1, b_2) \mid a_1(a_2^2 - b_2^2) - 2a_2b_1b_2 = 0 \text{ or } a_2(a_1^2 + b_1^2) - 2b_2(a_1 + b_1) = 0\}.$$

If $x_1 = y_1 = 0$, then $\lambda_3 = 0$ holds, and $\lambda_1 = \lambda_2 = 0$ can be modified to

$$(5.2) \quad a_2(b_1 + y_2) - b_2(a_1 + x_2) = 0, \quad x_2(a_1 + x_2) + y_2(b_1 + y_2) = 0.$$

If $(a_2, b_2) \neq (0, 0)$, since $a_1 + x_2 \neq 0$, we obtain

$$x_2 = \frac{b_2(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2}, \quad y_2 = \frac{a_2(-a_2b_1 + a_1b_2)}{a_2^2 + b_2^2}.$$

Since $a_1 + x_2 \neq 0$ and $b_1 + 2y_2 \neq 0$, we obtain a part of N :

$$N_2 = \{(a_1, a_2, b_1, b_2) \mid a_1a_2 + b_1b_2 = 0\}.$$

Next we assume $(b_1 + 2y_2, (a_1 + 2x_2)y_1) = (0, 0)$. If $(b_1 + 2y_2, a_1 + 2x_2) = (0, 0)$, then we have $x_2 = y_2 = 0$ by $\lambda_2 = 0$. Thus we see $a_1 = b_1 = 0$. In this case, $C = (0, 0, a_2, b_2) \in N_2$. We assume $(b_1 + 2y_2, y_1) = (0, 0)$, and $a_1 + 2x_2 \neq 0$. Then we have $x_1 = 0$ by $\lambda_3 = 0$, and $a_1x_2 + x_2^2 - y_2^2 = 0$ by $\lambda_2 = 0$. Then $\theta = a_2\eta_2 + (a_1 + x_2)\eta_3$ is a generator of kernel of $\mathcal{H}_{(\eta_1, \eta_2, \eta_3)}$. Then θH is a non-zero multiplication of $a_2(3a_1^2 + 7a_1x_2 + 4x_2^2 + 2y_2^2)$. Substituting $a_1x_2 + x_2^2 - y_2^2 = 0$ into this formula, we have $3a_2(a_1 + x_2)(a_1 + 2x_2) = 0$, which implies $a_2 = 0$. Then we have $b_2 = 0$ by $\lambda_1 = 0$. In this case, $C \in N_2$. On the other hand, we also see that if $C = (a_1, a_2, b_1, b_2) \in N$ and $q = (x_1, x_2, y_1, y_2) \in S(\tilde{L})$ satisfies the condition (ii), then $C \in N_1 \cup N_2$. Summarizing the above arguments, if $a_1 + x_2 \neq 0$, then we have a part of the non-cusp locus $N_1 \cup N_2$. By symmetry, we may interchange the subscript 1 with 2. Thus we obtain another part of N in the case of $a_2 + x_1 \neq 0$:

$$N_3 = \{(a_1, a_2, b_1, b_2) \mid a_2(a_1^2 - b_1^2) - 2a_1b_1b_2 = 0 \text{ or } a_1(a_2^2 + b_2^2) - 2b_1(a_2 + b_2) = 0\}.$$

Next, we assume $a_1 + x_2 = a_2 + x_1 = 0$ and $(b_1 + y_2, b_2 + y_1) \neq (0, 0)$. Then by the same method, we see $C \in N_1 \cup N_2 \cup N_3$. Also if $a_1 + x_2 = a_2 + x_1 = b_1 + y_2 = b_2 + y_1 = 0$, then we see $C \in N_1 \cup N_2 \cup N_3$. On the other hand, if $C \in N$ and $q \in S(\tilde{L})$ satisfies the condition (i), then we also see $C \in N_1 \cup N_2 \cup N_3$. Summarizing all these arguments, we have

$$\begin{aligned} N &= N_1 \cup N_2 \cup N_3 \\ &= \{C = (a_1, a_2, b_1, b_2) \mid a_1(a_2^2 - b_2^2) - 2a_2b_1b_2 = 0\} \\ &\quad \cup \{C \mid a_2(a_1^2 + b_1^2) - 2b_2(a_1 + b_1) = 0\} \cup \{C \mid a_1a_2 + b_1b_2 = 0\} \\ &\quad \cup \{C \mid a_2(a_1^2 - b_1^2) - 2a_1b_1b_2 = 0\} \cup \{C \mid a_1(a_2^2 + b_2^2) - 2b_1(a_2 + b_2) = 0\}. \end{aligned}$$

We draw the pictures of $N|_{b_2=-\varepsilon}$, $N|_{b_2=0}$ and $N|_{b_2=\varepsilon}$ in the (a_1, a_2, b_1) -space for small ε in Figure 1. Here, the thick line in $N|_{b_2=0}$ stands for the wrinkling.

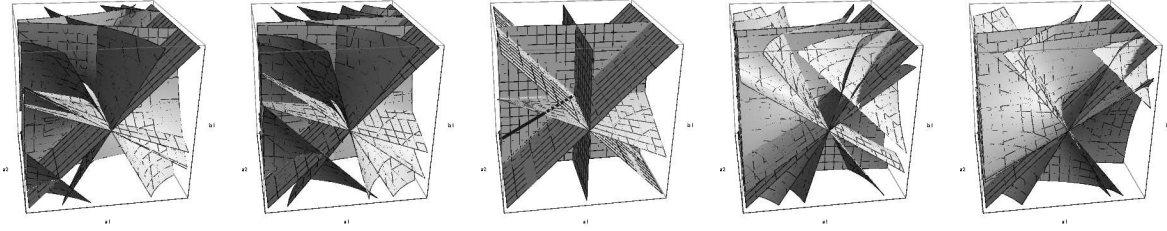


Figure 1: Non-cusp locus. $N|_{b_2=-1/2}$, $N|_{b_2=-1/4}$, $N|_{b_2=0}$, $N|_{b_2=1/4}$, $N|_{b_2=1/2}$

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